

A NOTE ON A COUNTEREXAMPLE OF DELGADO

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In this note we correct some incorrect analysis appearing in the paper of J. A. Delgado [1].

The example concerns two plane curves γ_1, γ_2 , which both are regular and complete, and have nonnegative curvature κ , i.e., $\kappa(\gamma_1) \geq 0, \kappa(\gamma_2) \geq 0$.

In this example Delgado intended to show that γ_1 and γ_2 are internally tangent at 0 and that $\kappa(\gamma_1(t)) \geq \kappa(\gamma_2(s))$ whenever $N_1(t) = N_2(s)$ where N_1 (resp. N_2) is the unit outward normal of γ_1 (resp. γ_2). He also showed that γ_1 is not contained in the convex region formed by γ_2 , thus showing that Blaschke's theorem does not apply to curves with nonnegative rather than positive curvature. However his analysis is incorrect. The example should go as follows:

$$\gamma_1(t) = (pt, t^4), \quad t \in \mathbf{R}, \quad p > 1,$$

$$\gamma_2(s) = \begin{cases} (s, (s-1)^4), & s \in \mathbf{R}, \quad s \geq 1, \\ (s, 0), & s \in \mathbf{R}, \quad |s| \leq 1, \\ (s, (s+1)^4), & s \in \mathbf{R}, \quad s \leq -1, \end{cases}$$

$$N_1(t) = \frac{1}{(p^2 + 16t^2)^{1/2}}(4t^3, -p),$$

$$N_2(s) = \begin{cases} \frac{1}{(1 + 16(s-1)^6)^{1/2}}(4(s-1)^3, -1), & \text{if } s \geq 1, \\ (0, -1), & \text{if } |s| \leq 1, \\ \frac{1}{(1 + 16(s+1)^6)^{1/2}}(4(s+1)^3, -1), & \text{if } s \leq -1. \end{cases}$$

Hence $N_1(t) = N_2(s)$ iff $s > 1$, and $t = \sqrt[3]{p}(s - 1)$, $-1 < s < 1$ and $t = 0$ or $s \leq -1$ and $t = \sqrt[3]{p}(s + 1)$. We have

$$\kappa(\gamma_1(t)) = \frac{12pt^2}{(p^2 + 16t^6)^{3/2}},$$

$$\kappa(\gamma_2(s)) = \begin{cases} \frac{12(s-1)^2}{(1 + 16(s-1)^6)^{3/2}}, & \text{if } s \geq 1, \\ 0, & \text{if } |s| \leq 1, \\ \frac{12(s+1)^2}{(1 + 16(s+1)^6)^{3/2}}, & \text{if } s \leq -1, \end{cases}$$

(and not as appeared in [1]). So in fact we have

$$\begin{aligned} \kappa(\gamma_1(0)) &= \kappa(\gamma_2(s)) = 0, \quad |s| \leq 1, \\ \kappa(\gamma_1(t)) &< \kappa(\gamma_2(s)) \quad \text{for } N_1(t) = N_2(s), t \neq 0 \end{aligned}$$

(and not $\kappa(\gamma_1(t)) \geq \kappa(\gamma_2(s))$ as appeared in [1]).

Hence it is no surprise that γ_1 eventually leaves the convex region formed by γ_2 . However, looking at the conjecture the other way around we should have that γ_2 lies in the convex region formed by γ_1 . In fact what we find is that in no neighborhood of the origin does it do so. Thus the conjecture fails rather strongly. The fact that γ_1 "cuts" γ_2 for points $t \neq 0$ is now irrelevant. This point is made even more clear by the fact that if $p = 1$ then $\gamma_1 \cap \gamma_2 = \{0\}$ and the example still works.

References

- [1] J. A. Delgado, *Blaschke's theorem for convex hypersurfaces*, J. Differential Geometry **14** (1979) 489-496.
- [2] M. P. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall, Englewood Cliffs, NJ, 1976.
- [3] J. A. Thorpe, *Elementary topics in differential geometry*, Undergraduate Texts in Math., Springer, New York, 1979.

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